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# The Transformational Decomposition (TD) Method for Compressible Fluid Flow Simulations

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In transient flow through porous media, the Partial Differential Equation (PDE) to be solved is

$$\nabla \cdot (k \nabla p) = C_T \frac{\partial p}{\partial t} + q, \quad (1)$$

where  $p$  is the pressure,  $k$  is the permeability,  $q$  is the source or sink flow rate,  $t$  is the time,  $C_T = \phi \mu C_L$ ,  $\phi$  is the porosity, and  $\mu$  and  $C_L$  are the fluid viscosity and compressibility, respectively. Equation (1) is solved numerically in all but the simplest problems. The basic concept of any numerical method is the substitution of a set of algebraic equations for the original PDE. Instead of solving for the continuous smooth function  $p(x, y, z, t)$ , the space domain  $(x, y, z)$  is subdivided into  $N_D$  subdomains, and the time  $t$  is discretized in  $N_T$  time steps;  $N_T$  sets of approximations  $\tilde{p}$  of the solution are obtained at the  $N_D$  predetermined points in space.

Despite their power and flexibility, numerical solutions have some serious drawbacks. Minimization of the error introduced by the numerical approximation of the spatial derivatives in the PDEs dictates the discretization of the space domain into a large number of subdomains at all of which solutions must be obtained (whether desired or not). This increases the execution time requirements and requires a large amount of computer memory. Accuracy and stability considerations necessitate a large number of small time steps between observation times; solutions must be obtained at all these intermediate times, increasing the execution times and the roundoff errors.

The Transformational Decomposition (TD) method is a new method that addresses the shortcomings of traditional numerical techniques. The major advantage of the TD method is that it requires no time discretization and a very coarse space discretization to yield an accurate, stable solution that is semianalytical in time and analytical in space.

## THE TD METHOD

There are two stages in the TD method, the Decomposition stage and the Reconstitution stage. In the Decomposition stage, the original PDE is decomposed by using successive levels of integral transforms. The first step in this stage involves the application of the Laplace transform to eliminate the time dependency of the original PDE. The

resulting equation is then subjected to successive Finite Cosine Transforms (FCT).

Each level of FCT eliminates one active dimension until single-point equations remain. The original PDE is thus decomposed into much simpler point algebraic equations, for which solutions are obtained in the transformed space. In the Reconstitution stage, solutions in space and time are obtained by applying successive levels of inverse transforms. The development of the TD method in one dimension presented here neglects gravity and considers  $\phi$  a constant. The application of the TD method to multidimensional systems with gravity and pressure-dependent  $\phi$ 's can be found in Moridis and McVay (1993).

## Step 1: The Laplace Transform of the PDE

The boundary conditions in the 1-D problem depicted in Figure 1 are

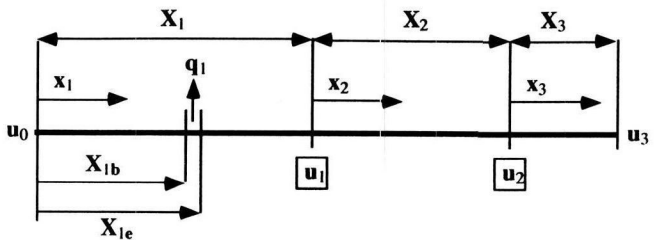
$$\begin{aligned} p'(x=0) = p'(x_1=0) = u_0 = 0, \\ p'(x=X_{\max}) = p'(x_3=X_3) = u_3 = 0, \end{aligned} \quad (2)$$

indicating no flow at the outermost boundaries, and

$$p'_i(x_1=X_1) = u_1(t), \quad p'_i(x_2=X_2) = u_2(t), \quad (3)$$

which describe internal boundaries that are unknown functions of time. The subscripted  $x_i$  ( $i = 1, 2, 3$ ) denote local coordinates; the global coordinates have no subscripts.

The Laplace transform of Eq. (1) in the  $i$ th ( $i = 1, 2, 3$ ) locally homogeneous subdomain of the 1-D problem yields



**Figure 1.** The TD method in one dimension. The quantities in the boxes indicate the unknown internal boundaries. [XBL 935-787]

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$$k_i \frac{d^2 \Psi_i}{dx_i^2} - C_T s \Psi_i = -C_T p_i(0) + \hat{q}_i, \quad (4)$$

where  $s$  is the Laplace parameter,  $p(0)$  is the distribution of  $p$  at  $t=0$ ,  $\Psi = L\{p\}$ ,  $\hat{q} = L\{q\}$ , and  $L\{\}$  denotes the Laplace transform of the quantity in the brackets. Equation (4) is subject to the boundary conditions

$$\Psi'_i(x_1=0) = U_0 = 0, \quad \Psi'_3(x_3 = X_3) = U_3 = 0, \quad (5)$$

$$\Psi'_1(x_1 = X_1) = U_1, \quad \Psi'_2(x_2 = 0) = \varepsilon_1 U_1, \quad (6)$$

$$\Psi'_2(x_2 = X_2) = U_2, \quad \Psi'_3(x_3 = 0) = \varepsilon_2 U_2,$$

where  $\Psi'_i = L\{p'_i = dp_i/dx_i\}$ ,  $U_\ell = L\{u_\ell(t)\}$  ( $\ell = 0, \dots, 3$ ), and  $\varepsilon_i = k_i/k_{i+1}$ . These boundary conditions incorporate the tangent law at the boundaries (continuity of fluxes).

## Step 2: The Finite Integral Transform

The FCT of Eq. (4) yields

$$k_i \left[ -\frac{n^2 \pi^2}{X_i^2} \Theta_i + (-1)^n \Psi'_{iX} - \Psi'_{i0} \right] - C_T s \Theta_i = -C_T \hat{p}_i(0) + \hat{q}_i, \quad (7)$$

where

$$\Theta_i(s, n) = \mathcal{F}_c\{\Psi_i\} = \int_0^{x_i} \Psi_i \cos\left(\frac{n\pi x_i}{X_i}\right) dx_i, \quad (8)$$

$\hat{p}_i(0) = \mathcal{F}_c\{p_i(0)\}$ ,  $\hat{q}_i = \mathcal{F}_c\{\hat{q}_i\}$ ,  $\Psi'_{i0} = \Psi'_i(x_i = 0)$ ,  $\Psi'_{iX} = \Psi'_i(x_i = X_i)$ , and  $\mathcal{F}_c\{\}$  denotes the FCT of the quantity in brackets. Note that  $p_i(0)$  need not be a constant; any known function of  $x_i$  for which an FCT exists is acceptable. As for the source/sink term  $\hat{q}_i$ , if  $\tilde{q} \neq 0$  for  $X_{ib} \leq x_i \leq X_{ie}$  (see Figure 1), then

$$\hat{q}_i = \begin{cases} \frac{\tilde{q} X_i}{n\pi} \left[ \sin\left(\frac{n\pi X_{ie}}{X_i}\right) - \sin\left(\frac{n\pi X_{ib}}{X_i}\right) \right] & n > 0 \\ \tilde{q}(X_{ie} - X_{ib}) & n = 0, \end{cases} \quad (9)$$

The flexibility that Eq. (9) affords is obvious, as it allows the positioning of wells anywhere in the subdomains. Equation (7) then yields

$$\Theta_i = \alpha_i \Psi'_{i,BO} + b_i \Psi'_{i,BX} + c_i, \quad (10)$$

where

$$a_i(s, n) = \frac{-r_i^2}{n^2 + \omega_i^2}, \quad b_i(s, n) = (-1)^{n+1} a_i, \quad (11)$$

$$c_i = c_i(s, n) = \frac{\omega_i^2}{n^2 + \omega_i^2} \frac{\hat{r}_i(0)}{s} - \frac{\hat{q}_i}{\lambda_i} \frac{\tau_i^2}{n^2 + \omega_i^2}, \quad (12)$$

and

$$\tau_i = \frac{X_i}{\pi}, \quad \omega_i = \tau_i \sigma_i, \quad \sigma_i = \sqrt{\frac{C_T s}{\lambda_i}}. \quad (13)$$

Equation (10) is a simple, single-point algebraic equation, and represents the decomposed form of the original 1-D PDE. If both  $\Psi'_{i,BO}$  and  $\Psi'_{i,BX}$  are known (as in the case of a homogeneous system), the decomposition stage ends here.

## Step 3: The Internal Boundary Conditions

In heterogeneous systems with multiple subdomains, the internal boundary conditions must be determined next. The pressures being equal on both sides of the boundary between subdomains 1 and 2 in Figure 1,  $\Psi_1(x_1 = X_1) = \Psi_2(x_2 = 0)$ . Applying the inverse FCT on the governing equations in the two subdomains gives

$$\begin{aligned} & \frac{1}{X_i} \Theta_1(s, 0) + \frac{2}{X_1} \sum_{n=1}^{\infty} \Theta_1(s, n) \cos\left(\frac{n\pi X_1}{X_i}\right) \\ &= \frac{1}{X_i} \Theta_2(s, 0) + \frac{2}{X_2} \sum_{n=1}^{\infty} \Theta_2(s, n) \cos\left(\frac{n\pi 0}{X_2}\right), \end{aligned} \quad (14)$$

from which we obtain the boundary equation

$$F_1 U_0 + G_1 U_1 + H_1 U_2 = R_1.$$

Closed forms for  $F$ ,  $G$ ,  $H$ , and  $R$  are obtained as

$$F_1 = -\frac{1}{\sigma_1 \sinh(\pi\omega_1)},$$

$$G_1 = \frac{1}{\sigma_1 \tanh(\pi\omega_1)} + \frac{\varepsilon_1}{\sigma_2 \tanh(\pi\omega_2)},$$

$$H_1 = -\frac{1}{\sigma_2 \sinh(\pi\omega_2)},$$

$$R_1 = \frac{r_{2c} - r_{1c}}{s} - \frac{\ddot{q}_2}{\lambda_2} \frac{1}{\sigma_2^2} \left[ \frac{\sinh(\omega_2 \pi - \sigma_2 X_{2b}) - \sinh(\omega_2 \pi - \sigma_2 X_{2e})}{\sinh(\omega_2 \pi)} \right] + \frac{\ddot{q}_1}{\lambda_1} \frac{1}{\sigma_1^2} \left[ \frac{\sinh(\sigma_1 X_{1e}) - \sinh(\sigma_1 X_{1b})}{\sinh(\omega_1 \pi)} \right]. \quad (19)$$

An analogous equation is obtained from subdomains 2 and 3. For any subdomain  $i$  other than the first, the numerator of  $F_i$  must be changed to  $\varepsilon_{i-1}$ . Since  $U_0 = U_3 = 0$ , the unknown  $U_1$  and  $U_2$  are readily obtained from the two boundary equations. In general, if there are  $N$  subdomains with locally homogeneous properties; these define  $N + 1$  boundaries of which two (the outermost) are known, and the remaining  $N_B = N - 1$  are the internal unknown boundaries. Writing the resulting  $N_B$  simultaneous equations in a matrix notation, we have

$$\mathbf{M}_v \vec{U}_v = \vec{R}_v \quad \text{and} \quad \vec{U}_v = \mathbf{M}_v^{-1} \vec{R}_v, \quad (20)$$

where  $\mathbf{M}$  is the coefficient matrix,  $\vec{R}$  is the “known” right-hand side vector, and  $\vec{U}$  is the vector of the unknown conditions  $U = L\{\Psi\}$  at the internal boundaries. Values for the Laplace parameter  $s$  for an observation time  $t$  are provided by the Stehfest (1970) algorithm as  $s_v = (\ln 2/t)^v, v = 1, \dots, N_S$ . Optimum values for  $N_S$  were discussed by Moridis et al. (1991). Steps 1 through 3 represent the Decomposition stage. The Reconstitution stage is described in Steps 4 and 5.

#### Step 4: The Laplace Domain Solution

Once the  $\vec{U}_v$  becomes known, then the  $\Psi_v$  at any point  $x_\ell$  within a subdomain  $i$  with boundaries  $U_\xi$  and  $U_{\xi+1}$  is given by the inverse FCT as

$$\Psi_i(s_v, x_\ell) = a_{iT} U_\xi(v) + b_{iT} U_{\xi+1}(v) + c_{iT}, \quad (21)$$

where

$$a_{iT} = \varepsilon_{i-1} F_1 \cosh(\omega_i \pi - \sigma_i x_\ell), \quad (22)$$

$$b_{iT} = -F_1 \cosh(\sigma_i x_\ell), \quad (23)$$

$$c_{iT} = \frac{\ddot{q}_i}{2\lambda_i} \frac{1}{\sigma_i^2} \left\{ \frac{\sinh[\omega_i \pi - \sigma_i (X_{ie} + x_\ell)]}{\sinh(\omega_i \pi)} + \frac{\sinh[\omega_i \pi - \sigma_i (X_{ie} + x_\ell)]}{\sinh(\omega_i \pi)} - \frac{\sinh[\omega_i \pi - \sigma_i (X_{ib} + x_\ell)]}{\sinh(\omega_i \pi)} - \frac{\sinh[\omega_i \pi - \sigma_i (X_{ib} + x_\ell)]}{\sinh(\omega_i \pi)} \right\} + \frac{r_{ic}}{s}. \quad (24)$$

#### Step 5: The Solution at Time $t$

To obtain a solution at a time  $t$  at a number of desired points  $x_\ell$ , all vectors  $\vec{\Psi}_v, v = 1, \dots, N_S$  are needed. Using the Stehfest algorithm, the unknown vector  $\vec{p}$  at time  $t$  is computed as

$$\vec{p}(t) = \frac{\ln 2}{t} \sum_{v=1}^{N_S} W_v \cdot \vec{\Psi}_v, \quad (25)$$

where the terms  $W_v$  are constant. Because of its formulation, the TD method provides semianalytical solutions that are fully differentiable and integrable; continuous velocity fields are thus easily determined, and mass balance calculations over the subdomains are simple.

#### VERIFICATION

The test problem used for verification involves flow to a well located at a vertical fracture of length  $L$  (perpendicular to the  $x$  axis) and depth  $h$  in a rectangular (1-D) reservoir. The TD solution is compared with FD solutions, as well as with the analytical solution given by

$$p_D = L^{-1}\{\Psi_D\}, \quad \Psi_D = H_1 + H_2, \quad (26)$$

where

$$H_1 = \frac{\exp(-x_D\sqrt{s})}{s^{1.5}[1 - \exp(-2X_D\sqrt{s})]}, \quad (27)$$

$$H_2 = \frac{\exp[(x_D - 2X_D)\sqrt{s}]}{s^{1.5}[1 - \exp(-2X_D\sqrt{s})]}, \quad (28)$$

$p_D = (p_i - p)(k\sqrt{A}/q\mu)$ ,  $t_D = (kt/A\phi\mu C_L)$ , and  $x_D = (x/\sqrt{A})$ . Here  $X$  is the length of the reservoir, and  $A = L \times h$ .

Two subproblems are investigated. In Problem 1a we obtain the TD solution using the equation for a single homogeneous domain (as specified). In Problem 1b we test the performance of the TD concept by subdividing the domain (5000 ft) into two subdomains (300 and 4200 ft) and comparing the solution with the one from Problem 1a. The two TD solutions are virtually identical, differing in the 8th or 9th decimal place. This confirmed the validity of the concept. All results correspond to both subproblems and are presented together.

Figure 2 shows the TD solutions at a number of times. The measure of the accuracy of the TD method is given by Figure 3, which compares the TD results with the analytical solution. The observed deviations are extremely small, and the TD method is shown to be practically insensitive to the size of the time increment (thus allowing an unlimited time step).

Figure 4 compares the TD solution at  $t_D = 1000$  (i.e., using a time step size of 533.35 days) with the FD solutions obtained for various space discretizations. To minimize the contribution of time-related truncation error to the FD solutions, a very fine time discretization is used, requiring 543 time steps and 1117 matrix inversions. With an increasingly fine space discretization, the FD solutions approach the TD solution. The superiority of the TD is obvious, as it is capable of delivering a more accurate solution with a single (or none at all) algebraic equation (which has to be solved  $N_S$  times) than an FD scheme, which needs to invert 1117 times a matrix of order 111. This reduces the computational effort by orders of magnitude.

In Figure 5 the TD method is compared with FD solutions with increasingly fine time discretizations. To minimize the effects of space-related truncation errors, a

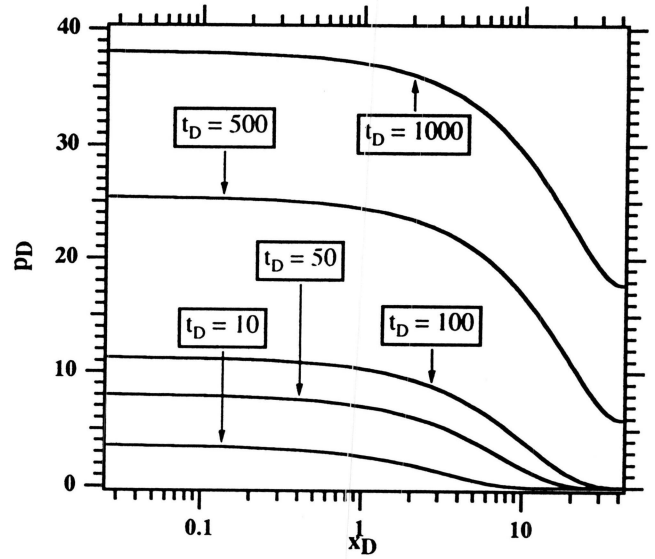


Figure 2. The TD solutions of  $p_D$  for various  $t_D$ 's. [XBL 935-788]

fine space discretization (111 gridblocks) is used. A pattern similar to the one observed in Figure 4 is evident: with an increasingly fine time discretization, the FD solutions tend toward the analytical solutions and the TD solutions, further attesting to the power of the method. The TD solution exhibits a very small deviation from the analytical solution and is consistently superior to the FD solution. The superiority of TD persisted even when a very fine time discretization (263 time steps, and a total of 541 matrix

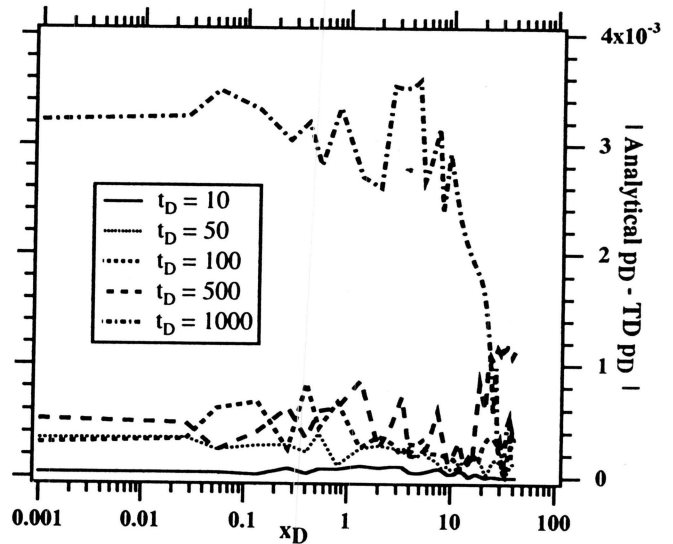


Figure 3. Absolute differences between the TD and the analytical solutions. [XBL 935-789]

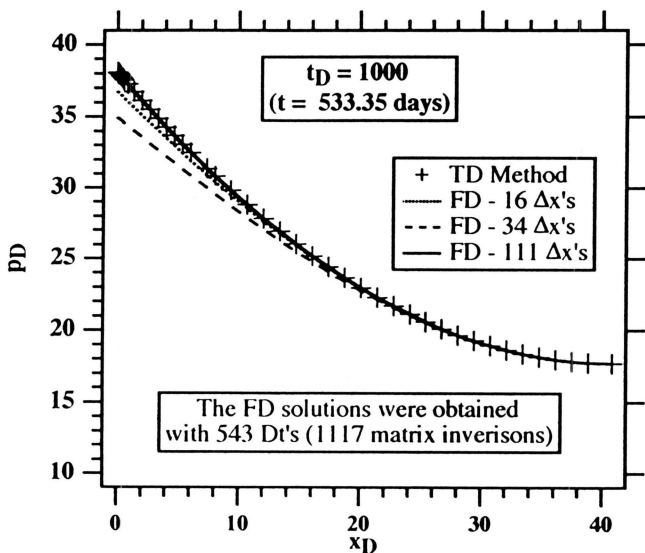


Figure 4. Comparison of the TD with the FD solution for various space discretizations. [XBL 935-790]

solutions) was used in the FD simulation. The corresponding TD computational effort to achieve this level of accuracy is essentially trivial: either direct substitution into Eqs. (22) through (24) at the desired time and location or (in the case of the two subdomains of subproblem 1b) solution of the single-point algebraic equation (20)  $N_S$  times, followed by the direct substitution.

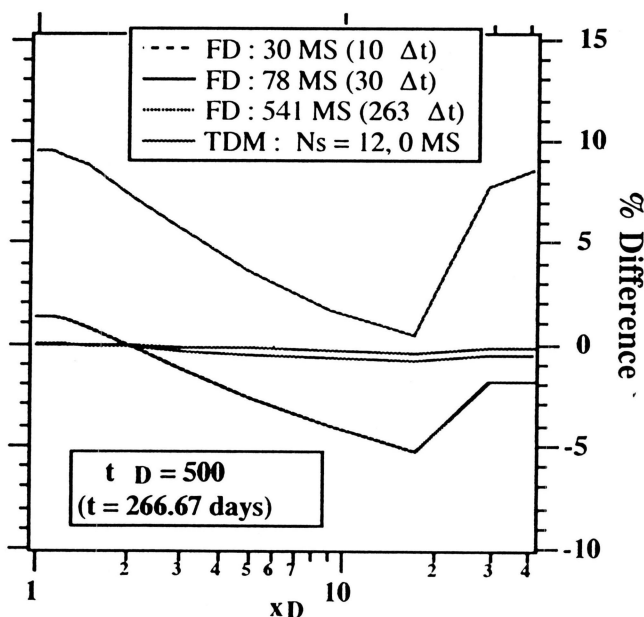


Figure 5. Comparison of the TD solution with the FD solutions for various time discretizations. [XBL 935-791]

## CONCLUSIONS

A new numerical method, the Transformational Decomposition (TD) method, was developed for the solution of the nonlinear, parabolic Partial Differential Equation (PDE) of transient, slightly compressible, single-phase liquid flow through porous media. Because TD uses a Laplace transform formulation, it eliminates the need for time discretization and allows an unlimited time step size without loss of stability or accuracy. By using Finite Cosine Transforms, the method drastically reduces the need for space discretization, requiring only a small number of large subdomains for an accurate solution. The TD method provides semianalytical solutions in space and time by decomposing the original PDE into a small number of algebraic equations and by equating and solving for conditions at internal boundaries. These solutions are fully differentiable and integrable, allowing the determination of continuous velocity maps and easy mass balance calculations.

With finer space and time discretizations, the FD solutions tend to the TD solution. The TD method provides a solution generally more accurate than the FD solution. This was expected because the elimination of the traditional time and space discretizations limit the truncation error.

The TD method may significantly reduce the computer memory requirements because discretization in time is not needed and a very coarse grid suffices for the space discretization. Execution times may be substantially reduced because smaller matrices are inverted in the TD method, and solutions are obtained at the desired points in space and time only, whereas in standard numerical methods solutions are necessary at all of the points of the discretized time and space domains.

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